Let $-\infty < a < b < \infty$.

**Proposition 5.12:**

Let $f \in C^2((a,b))$ with $f'' \geq 0$. Then we have for all $x_0 \neq x_1 \in (a,b)$, $0 < t < 1$ the inequality:

$$f(tx_1 + (1-t)x_0) \leq tf(x_1) + (1-t)f(x_0), \quad (*)$$

and the inequality is strict, if $f'' > 0$.

**Proof:**

i) Fix $x_0 \neq x_1 \in (a,b)$. Consider the function $g \in C^2([0,1])$:

$$g(t) = f(tx_1 + (1-t)x_0) - (tf(x_1) + (1-t)f(x_0))$$

with

$$g(0) = g(1) = 0,$$

$$g''(t) = f''(tx_1 + (1-t)x_0)(x_1 - x_0)^2 \geqslant 0, \quad 0 \leq t \leq 1.$$

Assume to the contrary,

$$\max_{0 \leq t \leq 1} g(t) = g(t_{max}) > 0, \text{ where } 0 < t_{max} < 1.$$

According to Corollary 5.3 we have

$$g'(t_{max}) = 0$$

$\Rightarrow$ Prop. 5.11 gives $\tau \in (t_{max}, 1)$ s.t.

$$0 = g(1) = g(t_{max}) + g'(t_{max})(1-t_{max}) + g''(\tau)\frac{(1-t_{max})^2}{2} \geq g(t_{max}) > 0,$$

$\Longrightarrow$ contradiction. Thus $g(t) \leq 0$ and the claim follows from the definition of $g$.

ii) Analogously, we obtain a contradiction if $f'' > 0$, if one assumes that $g(t_0) = 0$ for some $0 < t_0 < 1$.

$\square$

<u>Definition 5.5:</u>

A function $f: (a,b) \longrightarrow \mathbb{R}$ with the property (*) is called "convex"; and "strictly convex", if we have

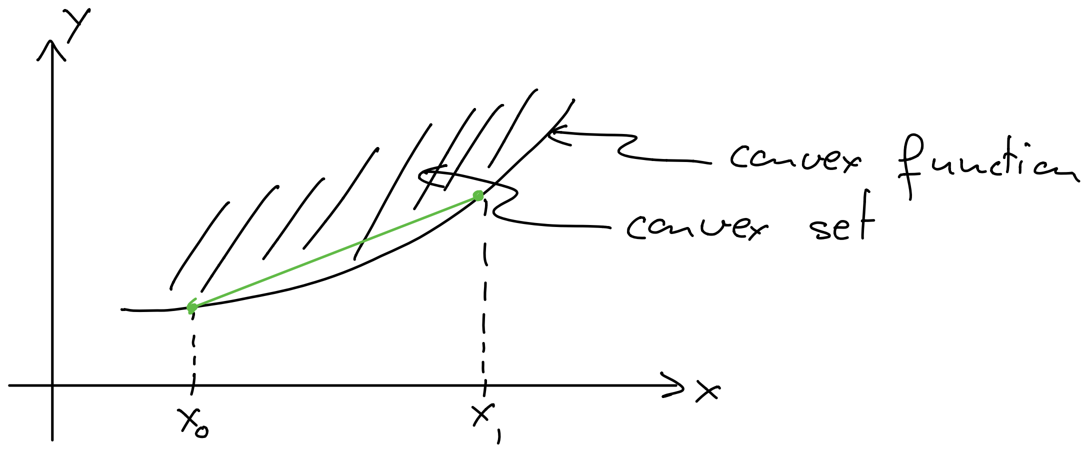
$$\forall\; x_0 \neq x_1 \in (a,b),\; 0 < t < 1:$$
$$f(tx_1 + (1-t)x_0) < tf(x_1) + (1-t)f(x_0).$$

<u>Remark 5.7:</u>

i) Aparently, $f$ is (strictly) convex if and only if the "Epigraph" of $f$,

$$epi(f) := \{(x,y) \mid x \in (a,b),\, y \geq f(x)\} \subset \mathbb{R}^2,$$

is a (strictly) convex set.




ii) Prop. 5.12 gives a convenient criterion for the (strict) convexity of $f$.

<u>Example 5.17:</u>

i) The function $f(x) = |x|$, $x \in \mathbb{R}$, is convex.

ii) If $f, g : (a,b) \longrightarrow \mathbb{R}$ are two convex functions, then $f+g : (a,b) \longrightarrow \mathbb{R}$ is also convex, with $f_k : (a,b) \longrightarrow \mathbb{R}$, $k \in \mathbb{N}$, also

$$f = \sum_{k=1}^{\infty} f_k$$

iii) $\exp'' = \exp > 0$, therefore, exp is strictly convex.

iv) Let $f(x) = x \log x$, $x > 0$. We have

$$f'(x) = \log x + 1, \quad f''(x) = \frac{1}{x} > 0,$$

therefore $f$ is strictly convex according to Prop. 5.12.

v) For a fixed number $\alpha > 1$, let
$f(x) = x^\alpha = \exp(\alpha \log x)$, $x > 0$. As

$$f'(x) = \alpha x^{\alpha-1}, \quad f''(x) = \alpha(\alpha-1)x^{\alpha-2} > 0,$$

$f$ is strictly convex according to Prop. 5.12.

The property (*) also holds for more than 2 points.

<u>Proposition 5.13</u> (Jensen):

Let $f:(a,b) \longrightarrow \mathbb{R}$ convex. Then we have for arbitrary points $x_1, \ldots, x_N \in (a,b)$ and numbers $0 \leq t_1, \ldots, t_N \leq 1$ with $\sum_{i=1}^N t_i = 1$ the inequality

$$f\left(\sum_{i=1}^N t_i x_i\right) \leq \sum_{i=1}^N t_i f(x_i)$$

<u>Proof</u> (by Induction):

<u>$N=1$:</u> ✓ (also $N=2$ according to definition)

<u>$N \longrightarrow N+1$:</u> Let $t_1 < 1$

set $x_0 = \sum_{i=2}^{N+1} \frac{t_i}{1-t_1} x_i$.

As $f$ is convex, we obtain

$$f\left(\sum_{i=1}^{N+1} t_i x_i\right) = f\left(t_1 x_1 + (1-t_1) x_0\right)$$

$$\leq t_1 f(x_1) + (1-t_1) f(x_0)$$

According to induction assumption we have

$$f(x_0) = f\left(\sum_{i=2}^{N+1} \frac{t_i}{1-t_1} x_i\right) \leq \sum_{i=2}^{N+1} \frac{t_i}{1-t_1} f(x_i)$$

and the claim follows.

$\square$

__Example 5.18:__

For all $0 < x_1, \ldots, x_n < \infty$, $0 \leq \alpha_1, \ldots, \alpha_n \leq 1$ with $\sum_{i=1}^{n} \alpha_i = 1$ we have

$$\prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i$$

__Proof:__

As the function exp is convex, the claim follows from Prop. 5.13:

$$\prod_{i=1}^{n} x_i^{\alpha_i} = \prod_{i=1}^{n} \exp(\alpha_i \log x_i)$$

$$= \exp\left(\sum_{i=1}^{n} \alpha_i \log x_i\right)$$

$$\overset{\text{Prop. 5.13}}{\leq} \sum_{i=1}^{n} \alpha_i \exp(\log x_i) = \sum_{i=1}^{n} \alpha_i x_i$$

$\square$

In particular, we obtain for $\alpha_i = \frac{1}{n}$, $1 \leq i \leq n$,

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n} \sum_{i=1}^{n} x_i \quad .$$

# §6. Numerical solution of equations

## §6.1 A fix-point theorem

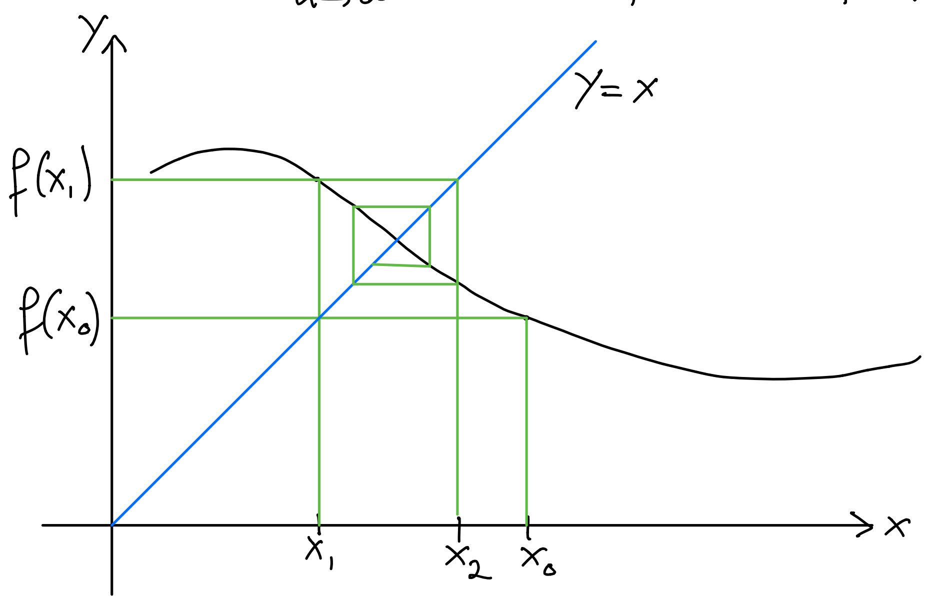
Often, one encounters the problem of solving an equation of the form $f(x) = x$, where $f: [a,b] \longrightarrow \mathbb{R}$ is a continuous function.

We can use the following approximation method:

$$x_n := f(x_{n-1}) \quad \text{for } n \geq 1$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is well-defined and converges against a $\xi \in [a,b]$, then $\xi$ is a solution of the above equation, as from continuity of $f$ we get

$$\xi = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(\xi)$$




The following Proposition covers an important case in which the above procedure converges:

<u>Proposition 6.1:</u>

Let $D \subset \mathbb{R}$ be a closed interval and $f: D \to \mathbb{R}$ a differentiable function with $f(D) \subset D$. Furthermore, $|f'(x)| \leq q$ for so $q < 1$ and all $x \in D$. Let $x_0 \in D$ be arbitrary and

$$x_n := f(x_{n-1}) \quad \text{for } n \geq 1.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ converges against a unique solution $\xi \in D$ of the equation $f(\xi) = \xi$.

The following error estimate holds

$$|\xi - x_n| \le \frac{q}{1-q}|x_n - x_{n-1}| \le \frac{q^n}{1-q}|x_1 - x_0|$$

Remark 6.1:

The method converges faster for smaller $q$.
Suppose we have to solve the equation $F(x)=0$, where $F$ is a differentiable function with continuous derivative.

$\rightarrow$ set $f(x) := x - \frac{1}{c}F(x)$, where $F'(x^*) =: c \neq 0$

Choose $x^*$ close to solution $\xi$ of $F(\xi)=0$
$\Leftrightarrow$ $f(\xi) = \xi$. We have $f'(x^*) = 0$, thus $|f'(x)|$ is small for $x$ sufficiently close to $x^*$.

Proof of Prop. 6.1:

i) From the mean value theorem we obtain

$$|f(x) - f(y)| \le q|x-y| \quad \text{for all } x,y \in D$$

$$\Rightarrow |x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le q|x_n - x_{n-1}|$$

and by induction over $n$ we get:

$$|x_{n+1} - x_n| \le q^n |x_1 - x_0| \quad \text{for } n \in \mathbb{N}.$$

As

$$x_{n+1} = x_0 + \sum_{k=0}^{n}(x_{k+1} - x_k),$$

and the sequence $\sum_{k=0}^{\infty}(x_{k+1} - x_k)$ converges (use for example quotient criterion), the limit

$$\xi := \lim_{n\to\infty} x_n$$

exists. Furthermore, as $D$ is closed, $\xi \in D$, and satisfies $\xi = f(\xi)$.

ii) "uniqueness":

Suppose $\eta$ is another solution of $\eta = f(\eta)$, then we get

$$|\xi - \eta| = |f(\xi) - f(\eta)| \leq q|\xi - \eta|,$$

and from $q < 1$ it follows that $|\xi - \eta| = 0$, therefore $\xi = \eta$.

iii) "error estimate":

For all $n \geq 1$ and $k \geq 1$ we have

$$|x_{n+k} - x_{n+k-1}| \leq q^k |x_n - x_{n-1}|.$$

As $\xi - x_n = \sum_{k=1}^{\infty}(x_{n+k} - x_{n+k-1})$, we get

$$|\xi - x_n| \leq \sum_{k=1}^{\infty} q^k |x_n - x_{n-1}| = \frac{q}{1-q}|x_n - x_{n-1}|$$

$$\leq \frac{q^n}{1-q}|x_1 - x_0|.$$

$\square$